

A DERHAM MODEL FOR CHEN-RUAN COHOMOLOGY RING OF ABELIAN ORBIFOLDS

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ABSTRACT. We present a deRham model for Chen-Ruan cohomology ring of abelian orbifolds. We introduce the notion of *twist factors* so that formally the stringy cohomology ring can be defined without going through pseudo-holomorphic orbifold curves. Thus our model can be viewed as the classical description of Chen-Ruan cohomology for abelian orbifolds. The model simplifies computation of Chen-Ruan cohomology ring. Using our model, we give a version of wall crossing formula.

1. INTRODUCTION

In this paper we present a deRham model for Chen-Ruan cohomology ring of abelian orbifolds. We introduce the notion of *twist factors* so that formally the stringy cohomology ring can be defined without going through pseudo-holomorphic orbifold curves. Thus our model can be viewed as the classical description of Chen-Ruan cohomology for abelian orbifolds. The model simplifies computation of Chen-Ruan cohomology ring and gives a version of wall crossing formula.

In their original papers [3] and [4], the authors studied the Gromov-Witten theory of orbifolds. The theory in [4] may be read as quantum cohomology ring theory of orbifolds, while that in [3], as a special case of [4], serves as cohomology ring theory which is the now well-known Chen-Ruan cohomology ring of stringy orbifolds. We briefly review their construction for stringy abelian orbifolds in §2.

The attempts of computing the Chen-Ruan cohomology ring structure is most successful for toric orbifolds and their hypersurfaces. The group structure of their Chen-Ruan cohomology is computed by M. Poddar [9], [10] and the Chen-Ruan ring structure for toric orbifolds is computed by Borisov et al.[2]. In [8], Parker et al computed the ring structure for the mirror quintic 3-fold. The difficulty of the computation in [8] stems from the fact that the Chen-Ruan cup product as defined in [3] requires the computation of obstruction bundles over the moduli spaces of orbifold ghost curves.

In §3, we propose a new formulation of Chen-Ruan cohomology for stringy abelian orbifolds. A deRham type theory is constructed with each cohomology class being represented by formal forms while the Chen-Ruan product is interpreted as “wedge product” of formal forms. One may think of this as a classical level construction of Chen-Ruan theory. One advantage of the classical description is that it simplifies computations. To illustrate this point, in §5 we work out the computation of Chen-Ruan cohomology ring structure for the mirror quintic 3-fold and verifies the computations in [8]. Unfortunately, so far we have not found a similar way to deal with general orbifolds.

Let G be a Lie group. The natural category for symplectic reduction with respect to Hamiltonian G action is the category of symplectic orbifolds. As in the ordinary cohomology theory, it's natural to ask how the Chen-Ruan cohomology (ring) structure changes when crossing a wall. For example, the problem was posed in [3]. Wall crossing have been studied by various authors for smooth cases. In §4, we treat the problem for Chen-Ruan orbifold cohomology when G is abelian. Our formulation leads to a natural extension of equivariant cohomology to $H_{G,CR}^*$ for torus action from which the surjectivity of the corresponding Kirwan map $\kappa : H_{G,CR}^* \rightarrow H_{CR}^*$ follows naturally. The main result is theorem 4.3, which is stated

later. It reduces the change of the Chen-Ruan cohomology (ring) structure to computation at the fixed points in the wall. In §5 we apply the wall crossing formula to the simple case of weighted projective spaces, verifying the computation in [5].

Representing the cohomology class by forms in our new formulation (see §3), we state the main theorem as following

Theorem 4.3 *Let $G = S^1$ and X be a Hamiltonian S^1 -manifold with moment map $\mu : X \rightarrow \mathbb{R}$. Suppose $0 \in \mathbb{R}$ is a singular value and $F_{j \in J}$ be the fixed point components in $\mu^{-1}(0)$. Let $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in H_{G, CR}^*(X)$ and $p, q \in \mathbb{R}$ be two regular values of μ such that $0 \in (p, q)$ is the only singular value. Denote $\alpha_p = \kappa_p(\tilde{\alpha})$ and so on, then we have*

$$\langle \alpha_q \cup \beta_q, \gamma_q \rangle - \langle \alpha_p \cup \beta_p, \gamma_p \rangle = \sum_j \int_{F_j} \frac{\tilde{i}_{(g_1)}(\tilde{\alpha}) \tilde{i}_{(g_2)}(\tilde{\beta}) \tilde{i}_{(g_3)}(\tilde{\gamma})}{e_G(N_{F_j})},$$

where \cup is the Chen-Ruan cup product, $\langle \cdot, \cdot \rangle$ is the Poincaré pairing, $e_G(N_{F_j})$ is the equivariant Euler class of the normal bundle of F_j in X and $\tilde{i}_{(g)}(\cdot)$ is the equivariant twisted form defined by \cdot and $g \in S^1$.

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2. CHEN-RUAN COHOMOLOGY THEORY FOR ABELIAN ORBIFOLDS

In this section we review the theory of Chen-Ruan orbifold cohomology in the case where all the local isotropy groups are finite abelian groups. We refer to [3], [4] and the excellent book [1] for details and general setup.

2.1. Abelian orbifolds. We recall briefly the language of groupoids and the definition of orbifolds in this language. A *groupoid* \mathcal{G} consists of the datum $(G_0, G_1; s, t, m, u, i)$ in the diagram:

$$G_1 \underset{s \times t}{\times} G_1 \xrightarrow{m} G_1 \xrightarrow{i} G_1 \xrightleftharpoons[t]{s} G_0 \xrightarrow{u} G_1,$$

where G_0 is the space of *objects* and G_1 is the space of *arrows*, with s and t being the *source* and *target* maps. The map m defines *composition* of two arrows while i gives the *inverse* arrow. The map u is the *unit* map, which is a two sided unit for the composition. The maps satisfies a set of natural axioms, such as $s(u(x)) = t(u(x)) = x$. We sometimes denote $i(g) = g^{-1}$ and $m(g, h) = gh$. The notion of *morphism* between groupoids $\phi : \mathcal{H} \rightarrow \mathcal{G}$ consists of smooth maps $\phi_0 : H_0 \rightarrow G_0$ and $\phi_1 : H_1 \rightarrow G_1$ so that they are compatible with all the structure maps. Certain morphisms between groupoids are defined to be *equivalences* and the *Morita equivalence* between \mathcal{G} and \mathcal{G}' is defined by the existence of a groupoid \mathcal{H} and the diagram $\mathcal{G}' \xleftarrow{\phi} \mathcal{H} \xrightarrow{\psi} \mathcal{G}$ where ϕ and ψ are equivalences.

An *orbifold groupoid* is defined to be a proper seperable étale Lie groupoid. It means that G_0 and G_1 are smooth Hausdorff manifolds and the structure maps are all smooth, with s, t being local diffeomorphisms, so that $(s, t) : G_1 \rightarrow G_0 \times G_0$ is proper. It follows that $G_x = (s, t)^{-1}(x, x)$ for $x \in G_0$ is a finite group and is defined to be the *isotropy* or *local group* at x . The *orbit space* $|\mathcal{G}|$ of \mathcal{G} is defined to be the quotient space of G_0 under the equivalence relation $x \sim y$ iff they are connected by an arrow, i.e. $\exists g \in G_1$ so that $s(g) = x$ and $t(g) = y$. The simplest example for such a groupoid is the *action groupoid* $G \ltimes M$ of a finite group G acting on a manifold M , where $(G \ltimes M)_0 = M$ and $(G \ltimes M)_1 = G \times M$. The source and target maps are given by $(s, t) : (g, p) \mapsto (p, g \circ p)$ and the rest of structure maps are obvious.

Suppose that $\phi : \mathcal{G} \rightarrow \mathcal{H}$ is an equivalence between orbifold groupoids, then induced map on the orbit spaces $|\phi| : |\mathcal{G}| \rightarrow |\mathcal{H}|$ is a homeomorphism. An *orbifold structure* on a paracompact Hausdorff space X is defined to be an orbifold groupoid \mathcal{G} with a homeomorphism $f : |\mathcal{G}| \rightarrow X$

and (\mathcal{G}, f) and (\mathcal{G}', f') are *equivalent* iff \mathcal{G} and \mathcal{G}' are Morita equivalent and the maps f and f' are compatible under the equivalence relation. Then an *orbifold* \mathcal{X} is defined to be a space X with an equivalent class of orbifold structures. An orbifold structure (\mathcal{G}, f) in such an equivalent class is a *presentation* of the orbifold \mathcal{X} . We note that a presentation can be chosen such that over each point $x \in \mathcal{X}$, there is a component \tilde{U} of G_0 , so that the restricted groupoid is isomorphic to an action groupoid $G_{\tilde{x}} \ltimes \tilde{U}$ where $\tilde{x} \mapsto x$ under the quotient map. Such component \tilde{U} is sometimes called an *orbifold chart* around x . We thus see that, as abstract groups, G_x is well defined for $x \in X$.

Definition 2.1. An orbifold \mathcal{X} is an abelian orbifold if the local groups G_x for all $x \in X$ are abelian.

2.2. Twisted sectors. We recall here the definition of twisted sectors in the language of groupoids. Let \mathcal{G} be an orbifold groupoid, then a *left \mathcal{G} -space* M is a manifold with the *anchor map* $\pi : M \rightarrow G_0$ and *action map* $\mu : G_1 \times_{s, \pi} M \rightarrow M$ satisfying the usual identities of an action:

$$\pi(\mu(g, p)) = t(g), \mu(u(x), p) = p \text{ and } \mu(g, \mu(h, p)) = \mu(m(g, h), p),$$

whenever the terms are well defined. Similar to the case of group actions, we may define the *action groupoid* $\mathcal{G} \ltimes M$ for the \mathcal{G} -space M with $(\mathcal{G} \ltimes M)_0 = M$ and $(\mathcal{G} \ltimes M)_1 = G_1 \times_{s, \pi} M$. The source and target maps are given by $(s, t) : (g, p) \mapsto (p, \mu(g, p))$ as in the case of group actions. A special case is to let $M = G_0$, then the action groupoid is the groupoid \mathcal{G} .

In the following, we fix a groupoid presentation \mathcal{G} of the orbifold \mathcal{X} and will abuse notation and use \mathcal{G} and \mathcal{X} interchangeably. An *orbifold morphism* is defined by a morphism between some groupoid presentations of the orbifolds. For any $x \in G_0$, it induces a map of isotropy groups $\phi_x : H_x \rightarrow G_x$. The morphism is called *representable* if the map ϕ_x is injective for all $x \in G_0$. A morphism $\phi : \mathcal{H} \rightarrow \mathcal{G}$ of orbifold groupoids is an *embedding* if ϕ_0 is an immersion, $|\phi|$ is proper and satisfies a local condition which amounts to saying that the map $|\phi|$ can be locally lifted as a smooth map between the coverings. (cf. definition 2.3 of the book [?]). Then the pair (\mathcal{H}, ϕ) is a *sub-groupoid* of \mathcal{G} and correspondingly, the orbifold \mathcal{Y} with underlying space $|\mathcal{H}|$ defined by \mathcal{H} is a *sub-orbifold* of \mathcal{X} . Sometimes, we abuse the notation and say that ϕ , or \mathcal{H} is a suborbifold. The *intersection* of two suborbifold \mathcal{H} and \mathcal{H}' of \mathcal{G} is defined to be the fibered product $\mathcal{H} \times_{\phi, \phi'} \mathcal{H}'$ and will be denoted as usual $\mathcal{H} \cap \mathcal{H}'$.

The groupoid of *twisted sectors* $\wedge \mathcal{G}$ and *k-multisectors* \mathcal{G}^k can be defined as action groupoid of certain left \mathcal{G} -space $\mathcal{S}_{\mathcal{G}}^k$ constructed naturally from \mathcal{G} :

$$\mathcal{S}_{\mathcal{G}}^k = \{(g_1, g_2, \dots, g_k) \in G_1^k | s(g_1) = t(g_1) = s(g_2) = t(g_2) = \dots = s(g_k) = t(g_k)\}.$$

The anchor map is $\pi_k : \mathcal{S}_{\mathcal{G}}^k \rightarrow G_0 : (g_1, g_2, \dots, g_k) \mapsto x = s(g_1)$ and the action map for $s(h) = \pi_k(g_1, g_2, \dots, g_k)$ is by conjugation $\mu_k(h, (g_1, g_2, \dots, g_k)) = (hg_1h^{-1}, hg_2h^{-1}, \dots, hg_kh^{-1})$. We note that when the orbifold is abelian, the action of $h \in G_x$ is trivial. In terms of the quotient orbifold, we have

$$\tilde{\mathcal{X}}^k := |\mathcal{G}^k| = \{(x, (g_1, \dots, g_k)_{G_x}) | x \in X, g_i \in G_x, i = 1, \dots, k\}.$$

Let $\mathcal{S}_o^k \subset \mathcal{S}_{\mathcal{G}}^k$ be the set of tuples so that $g_1g_2 \dots g_k = 1$, then it is a left \mathcal{G} -subspace and correspondingly defines a subgroupoid \mathcal{G}_o^k of \mathcal{G}^k .

Associated to the orbifold groupoid \mathcal{G} , we have the *skeletal groupoid* \mathcal{C} with C_i the discrete set of connected components of G_i , for $i = 0, 1$. The structure maps are induced from those of \mathcal{G} . Then \mathcal{C} acts on the set $C(\mathcal{S}_{\mathcal{G}}^k)$ of connected components of $\mathcal{S}_{\mathcal{G}}^k$. Let $T^k = |\mathcal{C} \ltimes C(\mathcal{S}_{\mathcal{G}}^k)|$ and $(\mathbf{g}) = ((g_1, \dots, g_k)_{G_x}) \in T^k$ the image of the component containing (g_1, \dots, g_k) with $s(g_1) = x$. Then T^k parametrizes the connected components of $\tilde{\mathcal{X}}^k$. We have the disjoint union of sub-groupoids

$$\mathcal{G}^k = \sqcup_{(\mathbf{g}) \in T^k} \mathcal{G}_{(\mathbf{g})},$$

and correspondingly $\tilde{\mathcal{X}}^k = \sqcup_{(\mathbf{g}) \in T^k} \mathcal{X}_{(\mathbf{g})}$. Analogously, let $T_o^k = |\mathcal{C} \ltimes C(\mathcal{S}_o^k)| \subset T^k$ and similarly define \mathcal{S}_o^k and $\tilde{\mathcal{X}}_o^k$. Let $\mathcal{S}_{(\mathbf{g})}$ be the preimage of $\mathcal{X}_{(\mathbf{g})}$ under the natural quotient map,

then it is a \mathcal{G} -subspace of $\mathcal{S}_{\mathcal{G}}^k$. The groupoid $\wedge \mathcal{G}$ is also called the *inertia groupoid* of \mathcal{G} and the corresponding orbifold $\wedge \mathcal{X}$ the *inertia orbifold* or *orbifold of twisted sectors* of \mathcal{X} . Correspondingly, $\mathcal{X}_{(\mathbf{g})}$ is a k -multisector or a *twisted sector* when $k = 1$. The sector $\mathcal{X}_{(1)}$ associated to the unit is called the *untwisted sector*. The multisectors are sub-orbifolds of \mathcal{X} , where the (union of) embedding(s) $\phi^k : \mathcal{G}^k \rightarrow \mathcal{G}$ is defined by $\phi_0^k = \pi_k$ and $\phi_1^k(h, (g_1, \dots, g_k)_{G_x}) = h$. There are natural maps among the k -multisectors, which are induced by \mathcal{G} -equivariant maps among the $\mathcal{S}_{\mathcal{G}}^k$'s. The first class is the *evaluation maps*, induced by

$$e_{i_1, \dots, i_j} : \mathcal{S}_{\mathcal{G}}^k \rightarrow \mathcal{S}_{\mathcal{G}}^j : (g_1, g_2, \dots, g_k) \mapsto (g_{i_1}, g_{i_2}, \dots, g_{i_j}),$$

and the second class is the *involutions*, induced by

$$I : \mathcal{S}_{\mathcal{G}}^k \rightarrow \mathcal{S}_{\mathcal{G}}^k : (g_1, \dots, g_k) \mapsto (g_k^{-1}, \dots, g_1^{-1}).$$

It's easy to see that the evaluation maps are (unions of) embeddings and the involutions are isomorphisms. In particular, the evaluation map $e = \pi_k : \mathcal{S}_{\mathcal{G}}^k \rightarrow G_0$ induces an embedding of k -multisectors as sub-orbifold (groupoid) of \mathcal{G} .

An *orbifold bundle* \mathcal{E} over \mathcal{G} is by definition a \mathcal{G} -space so that $\pi : E \rightarrow G_0$ is a vector bundle and the action of \mathcal{G} on E is fiberwise linear, i.e. $g \in G_1$ induces linear isomorphism $g : E_{s(g)} \rightarrow E_{t(g)}$. The *total space* $|\mathcal{E}|$ of \mathcal{E} is given by the action groupoid $\mathcal{G} \ltimes E$ so that the projection morphism is defined by $\tilde{\pi} = (\pi, \pi_1)$ where $\pi_1 : G_1 \times_{\pi} E \rightarrow G_1$ is the projection to the first factor. Then the map $|\tilde{\pi}| : |\mathcal{E}| \rightarrow |\mathcal{G}| = X$ gives the corresponding orbibundle. A *section* σ of bundle \mathcal{E} is defined to be a \mathcal{G} -equivariant section of E , i.e. $s^* \sigma = t^* \sigma$ over G_1 . One example of orbifold bundle is the tangent bundle $T\mathcal{G}$, where the \mathcal{G} -space is TG_0 with the natural \mathcal{G} -action. The *pull-back* $\phi^* \mathcal{E}$ of \mathcal{E} by a morphism $\phi : \mathcal{H} \rightarrow \mathcal{G}$ is well-defined and is an orbifold bundle over \mathcal{H} . Now suppose that $\phi : \mathcal{H} \rightarrow \mathcal{G}$ is an oriented suborbifold groupoid, then we define the *normal bundle* of \mathcal{H} in \mathcal{G} as the quotient $N_{\mathcal{H}|\mathcal{G}} = \phi^* T\mathcal{G} / T\mathcal{H}$.

2.3. Degree shifting. From now on, we assume that \mathcal{X} is almost complex, that is, there is an almost complex structure on the tangent bundle TG_0 , which is invariant under the \mathcal{G} -action. It follows that the k -multisectors have induced almost complex structures as well. The *singular cohomology* $H^*(\mathcal{X})$ of an orbifold \mathcal{X} is defined to be the singular cohomology $H^*(X)$ of the underlying space $X = |\mathcal{X}|$. As ungraded group, the *Chen-Ruan orbifold cohomology group* $H_{CR}^*(\mathcal{X})$ of \mathcal{X} is defined to be

$$H_{CR}^*(\mathcal{X}) = H^*(\wedge \mathcal{X}) = \oplus_{(g) \in T_1} H^*(\mathcal{X}_{(g)}).$$

We now explain the grading. The degree of elements in $H^*(\mathcal{X}_{(g)})$ as elements in $H_{CR}^*(\mathcal{X})$ is different from their degree in $H^*(\mathcal{X}_{(g)})$. The difference is the *degree shifting number* $\iota_{(g)}$, which is defined below.

For $x \in \mathcal{X}_{(g)}$, let $g \in \mathcal{S}_{\mathcal{G}}$ be a preimage of x and $\tilde{x} \in G_0$ be the image of g under the evaluation map e . Then $g \in G_{\tilde{x}}$ and we have the decomposition into eigenspaces of g action:

$$e^* T_{\tilde{x}} G_0 = T_g \mathcal{S}_{\mathcal{G}} \oplus N_g = T_g \mathcal{S}_{\mathcal{G}} \oplus \oplus_{j=1}^m E_{j,g},$$

where g action is trivial on the first summand and non-trivial on the rest. Choose a (complex) basis according to the above decomposition. Then the g action can be represented by a diagonal matrix

$$(2.1) \quad \text{diag}(1, \dots, 1, e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_m}), \text{ where } \theta_j \in \mathbb{Q} \cap [0, 1) \text{ for all } j.$$

The number $\iota(x, g) = \sum_j \theta_j$ doesn't depend on $x \in \mathcal{X}_{(g)}$ and is defined to be the *degree shifting number* $\iota_{(g)}$ for the twisted sector $\mathcal{X}_{(g)}$. In fact, the decompositions for all $x \in \mathcal{X}_{(g)}$ fit together and give a decomposition of tangent bundle with respect to (g) action:

$$(2.2) \quad e^* T\mathcal{G} = T\mathcal{X}_{(g)} \oplus N_{(g)} = T\mathcal{X}_{(g)} \oplus \oplus_{j=1}^m E_j.$$

We assume that each E_j has rank 2 where $2m$ is the codimension of $\mathcal{X}_{(g)}$. (In general, E_j may not be a (complex) line bundle, in which case we may use standard splitting principle to proceed in the later arguments.) It's obvious that the bundle $N_{(g)}$ may be taken as the normal bundle $N_{\mathcal{X}_{(g)}|\mathcal{X}}$ of $\mathcal{X}_{(g)}$ in \mathcal{X} .

Under decomposition (2.2), the matrix representing g^{-1} can also be diagonalized:

$$\text{diag}(1, \dots, 1, e^{2\pi i \theta'_1}, \dots, e^{2\pi i \theta'_m}), \text{ where } \theta'_j \in \mathbb{Q} \cap [0, 1) \text{ for all } j.$$

Then it's easy to see that

$$(2.3) \quad \theta'_j + \theta_j = 1 \Rightarrow \iota_{(g^{-1})} + \iota_{(g)} = m.$$

Using the degree shifting number for the twisted sectors, we can write down $H_{CR}^*(\mathcal{X})$ in graded pieces:

$$H_{CR}^d(\mathcal{X}) = \oplus_{(g)} H^d(\mathcal{X}_{(g)})[-2\iota_{(g)}] = \oplus_{(g)} H^{d-2\iota_{(g)}}(\mathcal{X}_{(g)}).$$

Note that, in general, the grading is rational instead of integral.

2.4. Poincaré duality. The Poincaré duality holds in *de Rham* orbifold cohomology, which is isomorphic to the singular cohomology. The de Rham cohomology is defined as in the case of manifold, while the differential forms on an orbifold is defined to the sections of the orbifold bundle $\wedge^* T^* \mathcal{X}$. Let $\alpha \in \Omega^*(\mathcal{X})$ be a differential form, then the *support* $\text{supp}(\alpha)$ of α in \mathcal{X} is the image in X of its support in G_0 . Let $U \subset G_0$ be an orbifold chart and let α be a differential form with compact support in U . Then the integration is defined by:

$$(2.4) \quad \int_U^{orb} \alpha = \frac{1}{|G_U|} \int_U \pi^* \alpha.$$

Integration of general forms is then defined by partition of unity. For orbifolds admitting good covers, the pairing $\int_{\mathcal{X}}^{orb} \alpha_1 \wedge \alpha_2$ is a non-degenerate pairing between $H^*(\mathcal{X})$ and $H_c^*(\mathcal{X})$.

Poincaré duality holds in $H_{CR}^*(\mathcal{X})$ with the involutions

$$I : \mathcal{X}_{(g)} \rightarrow \mathcal{X}_{(g^{-1})}.$$

The pairing between $H_{CR}^d(\mathcal{X})$ and $H_{CR,c}^{2n-d}(\mathcal{X})$ is defined as the direct sum of the pairings on the twisted sectors $\mathcal{X}_{(g)}$ and $\mathcal{X}_{(g^{-1})}$:

$$\langle, \rangle^{(g)} : H^{d-2\iota_{(g)}}(\mathcal{X}_{(g)}) \times H_c^{2n-d-2\iota_{(g^{-1})}}(\mathcal{X}_{(g^{-1})}) \rightarrow \mathbb{R} : \langle \alpha, \beta \rangle = \int_{\mathcal{X}_{(g)}}^{orb} \alpha \wedge I^*(\beta).$$

Note that $d - 2\iota_{(g)} + 2n - d - 2\iota_{(g^{-1})} = 2(n - m)$ is the dimension of $\mathcal{X}_{(g)}$. The pairing $\langle, \rangle^{(g)}$ is simply the ordinary Poincaré duality on the abstract orbifold $\mathcal{X}_{(g)} \simeq \mathcal{X}_{(g^{-1})}$, which is non-degenerate.

2.5. Obstruction bundles. An important ingredient in defining Chen-Ruan orbifold cup product is the *obstruction bundles* on certain 3-multisectors (also called *triple twisted sectors*). Let $(\mathbf{g}) = ((g_1, g_2, g_3)) \in T_o^3$ and $E_{(\mathbf{g})} \rightarrow \mathcal{X}_{(\mathbf{g})}$ denote the obstruction bundle which we'll now describe.

Suppose that $\mathbf{r} = (r_1, r_2, r_3)$ records the orders of g_i and let $(S^2, \mathbf{z}, \mathbf{r})$ be an orbifold S^2 with 3 orbifold points $\mathbf{z} = (z_1, z_2, z_3)$. The local group at z_i is the cyclic group of order r_i for $i = 1, 2, 3$. Without loss of generality, we may assume that $\mathbf{z} = (0, 1, \infty)$ and drop \mathbf{z} from the notation.

Fix an almost complex structure J on the orbifold \mathcal{X} . We consider the space of representable pseudo-holomorphic orbifold morphisms $f : (S^2, \mathbf{r}) \rightarrow \mathcal{X}$, i.e. the local groups at z_i are mapped injectively to the local groups of the image. In particular, we are interested in the maps where $[f] = 0 \in H_2(X)$, i.e. constant maps. The moduli space of such constant maps is given by $\mathcal{X}_o^3 = \sqcup_{(\mathbf{g}) \in T_o^3} \mathcal{X}_{(\mathbf{g})}$, which contains $\mathcal{X}_{(\mathbf{g})}$ as a connected component. The evaluation maps $e_i : \mathcal{X}_{(\mathbf{g})} \rightarrow \mathcal{X}$ for $i = 1, 2, 3$ play the same role as the usual evaluation maps on marked points.

Let $y = f(S^2) \in \mathcal{X}_{(\mathbf{g})}$, $e : \mathcal{X}_{(\mathbf{g})} \rightarrow \mathcal{X}$ the evaluation map and consider the elliptic complex

$$(2.5) \quad \bar{\partial}_y : \Omega^0((e \circ f)^* T\mathcal{X}) \rightarrow \Omega^{0,1}((e \circ f)^* T\mathcal{X}),$$

which forms a family parameterized by $y \in \mathcal{X}_{(\mathbf{g})}$. The kernel of the family of elliptic complexes $\bar{\partial}_y$ is isomorphic to the bundle $T\mathcal{X}_{(\mathbf{g})}$ and the obstruction bundle $E_{(\mathbf{g})}$ is defined to be the cokernel.

More precisely, let $\langle \mathbf{g} \rangle$ be the subgroup of G_y generated by $\{g_i\}_{i=1}^3$. It can be shown that $\langle \mathbf{g} \rangle$, as abstract group, is independent of $y \in \mathcal{X}_{(\mathbf{g})}$. Since $(\mathbf{g}) \in T_o^3$, there is a branched covering $\phi : \Sigma \rightarrow S^2$ from a smooth compact Riemann surface Σ with covering group $\langle \mathbf{g} \rangle$ and branching loci over $(0, 1, \infty)$. The map $f : (S^2, \mathbf{r}) \rightarrow \mathcal{X}_{(\mathbf{g})}$ can then be lifted to the constant map $\tilde{f} : \Sigma \rightarrow \tilde{y} \in \mathcal{S}_{(\mathbf{g})}$ where $\tilde{y} \mapsto y$ under the quotient. Then the complex (2.5) lifts as the $\langle \mathbf{g} \rangle$ -invariant part of the following complex

$$(2.6) \quad \bar{\partial}_{\tilde{y}} : \Omega^0((e \circ \tilde{f})^*TG_0) \rightarrow \Omega^{0,1}((e \circ \tilde{f})^*TG_0),$$

where $e : \mathcal{S}_{(\mathbf{g})} \rightarrow G_0$ is the evaluation map. Since $(e \circ \tilde{f})^*TG_0 = \Sigma \times T_{e(\tilde{y})}G_0$ is a trivial bundle, it follows that

$$\text{coker} \bar{\partial}_{\tilde{y}} = H^{0,1}(\Sigma) \otimes T_{e(\tilde{y})}G_0.$$

We see that the cokernel above fits together to give an orbifold bundle $H^{0,1}(\Sigma) \otimes e^*T\mathcal{G}$ over $\mathcal{X}_{(g)}$. The group $\langle \mathbf{g} \rangle$ acts on the bundle with induced action on both factors. In particular, when the orbifold is abelian, the action of G_y on the cokernel is given by the induced action on $T_{\tilde{y}}G_0$ and trivial action on $H^{0,1}(\Sigma)$.¹ Then the obstruction bundle is

$$E_{(\mathbf{g})} = (H^{0,1}(\Sigma) \otimes e^*T\mathcal{X})^{(\mathbf{g})}, \text{ where } e : \mathcal{X}_{(\mathbf{g})} \rightarrow \mathcal{X} \text{ is the evaluation map.}$$

2.6. Chen-Ruan orbifold cup product. The definition of the Chen-Ruan orbifold cup product is the following. Let $(\mathbf{g}) \in T_o^3$ and $e_i : \mathcal{X}_{(\mathbf{g})} \rightarrow \mathcal{X}_{(g_i)}$ be the evaluation maps for $i = 1, 2, 3$. For $\alpha \in H^*(\mathcal{X}_{(g_1)})$, $\beta \in H^*(\mathcal{X}_{(g_2)})$ and $\gamma \in H_c^*(\mathcal{X}_{(g_3)})$, then $\alpha, \beta \in H_{CR}^*(\mathcal{X})$ and $\gamma \in H_{CR,c}^*(\mathcal{X})$, we define *3-point function*

$$\langle \alpha, \beta, \gamma \rangle = \int_{\mathcal{X}_{(\mathbf{g})}}^{orb} e_1^*(\alpha) e_2^*(\beta) e_3^*(\gamma) e(E_{(\mathbf{g})}),$$

where $e(E_{(\mathbf{g})})$ is the Euler class of the obstruction bundle $E_{(\mathbf{g})}$ (computed by choosing a connection while the integral does not depend on the choice). The Chen-Ruan cup product $\alpha \cup \beta \in H^*(\mathcal{X}_{(g_3^{-1})}) \subset H_{CR}^*(\mathcal{X})$ is then defined by Poincaré duality

$$(2.7) \quad \langle \alpha \cup \beta, \gamma \rangle = \langle \alpha, \beta, \gamma \rangle \text{ for all } \gamma \in H_c^*(\mathcal{X}_{(g_3)}),$$

It turns out that “ \cup ” defines an associative ring structure on $H_{CR}^*(\mathcal{X})$.

3. DERHAM MODEL OF $H_{CR}^*(\mathcal{X})$

It is well known from deRham theory that for a manifold X , the cohomology classes in $H^*(X)$ can be represented by closed forms. In this section, we will present a similar model for $H_{CR}^*(\mathcal{X})$ for abelian orbifold \mathcal{X} . Each cohomology class of (rational) degree d will be represented by a formal d -form. A natural “wedge product” can be defined for these formal forms. We will show that this wedge product can be identified with Chen-Ruan orbifold cup product. This somehow avoids the mysterious obstruction bundle.

3.1. Twist factors. To represent classes in $H^d(\mathcal{X}_{(g)})[-2\iota_{(g)}]$, besides a closed form on $\mathcal{X}_{(g)}$ we introduce an auxiliary term to account for the degree shifting. The auxiliary term works as “fractional Thom form”. We first recall the construction of Thom class in the category of orbifolds. Let $\pi : \mathcal{E} \rightarrow \mathcal{X}$ be an oriented orbifold vector bundle, then a *Thom form* Θ of \mathcal{E} is defined by a \mathcal{G} -invariant Thom form of the vector bundle $E \rightarrow G_0$. As in the case of manifold, Θ is compactly supported on \mathcal{E} near the 0-section and we have for $\alpha \in \Omega^*(\mathcal{E})$

$$(3.1) \quad \int_{\mathcal{E}}^{orb} \alpha \wedge \Theta = \int_{\mathcal{X}}^{orb} i^* \alpha, \text{ where } i : \mathcal{X} \rightarrow \mathcal{E} \text{ is the 0-section.}$$

¹In general, we should only take the action by $C(\mathbf{g})$, the elements in G_y that commute with \mathbf{g} . Here G_y is abelian, $C(\mathbf{g}) = G_y$.

The class represented by Θ is then the *Thom class* of \mathcal{E} and we denote it by $[\Theta]$.

Consider the splitting (2.2) and let l_j be a Thom form of E_j for $j = 1, \dots, m$. Then the Thom class of $N_{(g)}$ is given by $\prod_{j=1}^m [l_j]$. We have

Definition 3.1. The twist factor $t(g)$ of $\mathcal{X}_{(g)}$ is defined by the formal product

$$(3.2) \quad [t(g)] = \prod_{j=1}^m [l_j]^{\theta_j},$$

where θ_j is as given in (2.1).

Because of the similarity of (3.2) with the Thom class, $[t(g)]$ may be regarded as *fractional Poincaré dual* of $\mathcal{X}_{(g)}$ in \mathcal{X} . Formally, $t(g)$ is a form of degree $2\iota_{(g)}$ supported in a neighbourhood of $\mathcal{X}_{(g)}$ in \mathcal{X} . This formal degree makes up the difference between the degrees of the classes in $H_{CR}^*(\mathcal{X})$ and $H^*(\mathcal{X}_{(g)})$. We can then write the identification of $H^*(\mathcal{X}_{(g)})$ as a summand in $H_{CR}^*(\mathcal{X})$ formally as a *Thom isomorphism*. More precisely, suppose U be a neighbourhood of $\mathcal{X}_{(g)}$ in its normal bundle and identify it with a neighbourhood of $\mathcal{X}_{(g)}$ in \mathcal{X} , with the projection map $\pi : U \rightarrow \mathcal{X}_{(g)}$ and by representing cohomology classes by forms, we formally write

$$(3.3) \quad i_{(g)} : H^*(\mathcal{X}_{(g)}) \rightarrow H_{CR}^{*+2\iota_{(g)}}(\mathcal{X}) : [\alpha] \rightarrow [\pi^*(\alpha)t(g)].$$

We shall call $i_{(g)}(\alpha)$ a *twisted form* and note that it is supported in the neighbourhood U of $\mathcal{X}_{(g)}$. We'll drop π^* and simply write $i_{(g)}(\alpha) = \alpha t(g)$ since there should be no confusion.

Remark 3.1. In what follows, we'll carry out the (formal) computations using notations $[l_j]$ instead of l_j to emphasis that the results in cohomology do not depend on the particular choice of the forms l_j .

3.2. Poincaré duality. We will discuss the wedge product of two twisted forms later. As a warm up, we explain how the Poincaré duality (§2.4) follows from this formulation. We follow the convention that integration of the form $\int_{\mathcal{X}}^{orb} \alpha \wedge \prod_{j=1}^k t(g_j)$ vanishes unless the product $\prod_{j=1}^k [t(g_j)]$ gives the Thom class of some suborbifold \mathcal{Y} of \mathcal{X} , in which case (3.1) applies. Let $a = i_{(g)}(\alpha)$ and $b = i_{(g^{-1})}(\beta)$ and define pairing as

$$\langle a, b \rangle = \int_{\mathcal{X}}^{orb} a \wedge b = \int_{\mathcal{X}}^{orb} \alpha \wedge \beta \wedge t(g) \wedge t(g^{-1}) = \int_{\mathcal{X}_{(g)}}^{orb} \alpha \wedge \beta.$$

For the last equality, we use the fact that $t(g) \wedge t(g^{-1})$ is the Thom form of $\mathcal{X}_{(g)}$ in \mathcal{X} (cf. (2.3), (3.1)). This matches $\langle \alpha, \beta \rangle^{(g)}$ as defined in §2.4.

3.3. Wedge product. Now we assume that the orbifold \mathcal{X} is abelian. Then the normal bundle $N_{\mathcal{X}_{(\mathbf{g})}|\mathcal{X}}$ of any k -multisector $\mathcal{X}_{(\mathbf{g})}$ can be decomposed into direct sum of (complex) line bundles with respect to the $\langle \mathbf{g} \rangle$ -action (upto splitting principle):

$$(3.4) \quad N_{\mathcal{X}_{(\mathbf{g})}|\mathcal{X}} = \oplus_j E_j.$$

Let $a_i \in H_{CR}^{d_i+2\iota_{(g_i)}}(\mathcal{X})$, $i = 1, 2$, be two twisted forms. The wedge product $a_3 = a_1 \wedge a_2$ can be defined formally in the obvious way. We explain that a_3 is also a twisted form in a very natural way.

Proposition (Definition) 3.2. Suppose $a_i = i_{(g_i)}\alpha_i = \alpha_i t(g_i)$, for $i = 1, 2$ and define $a_3 := a_1 \wedge a_2 = i_{(g_1)}(\alpha_1) \wedge i_{(g_2)}(\alpha_2)$. Then $\exists \alpha_{3,j} \in H^*(\mathcal{X}_{(g_{3,j})})$ where $(g_1, g_2, g_{3,j}^{-1}) \in T_o^3$, so that $a_3 = \sum_j \alpha_{3,j} t(g_{3,j}) \in H_{CR}^*(\mathcal{X})$.

Proof. The formula defining a_3 is interpreted as following. The form $t(g_i)$, and thus a_i , is supported near $\mathcal{X}_{(g_i)}$ for $i = 1, 2$. It follows that a_3 is supported near $\mathcal{Z} = \mathcal{X}_{(g_1)} \cap \mathcal{X}_{(g_2)}$. We see that \mathcal{Z} is a union of 2-multisectors $\mathcal{Z}_j := \mathcal{X}_{(h_{1,j}, h_{2,j})}$ so that $h_{i,j}$ is in (g_i) for all j .

Let $g_{3,j} = h_{1,j}h_{2,j}$, then $((h_{1,j}, h_{2,j}, g_{3,j}^{-1})) \in T_o^3$ and we have $\mathcal{Z}_j \xrightarrow{i_{3,j}} \mathcal{X}_{(g_{3,j})}$ is naturally an embedding. We define

$$(3.5) \quad a_{3,j} = (a_1 \wedge a_2)_j := (i_{1,j}^*(\alpha_1) \wedge i_{2,j}^*(\alpha_2)) \wedge t(g_1) \wedge t(g_2),$$

where $i_{\bullet,j}$ is the inclusion $\mathcal{Z}_j \rightarrow \mathcal{X}_{(g_\bullet)}$. Then it's obvious that $a_3 = \sum_j a_{3,j}$. Thus we only need to prove the proposition for the case where \mathcal{Z} has only one component. The main issue is to deal with $t(g_1) \wedge t(g_2)$.

Assume that \mathcal{Z} has only one component. It is clear that the normal bundle $N_{\mathcal{Z}|\mathcal{X}}$ of \mathcal{Z} in \mathcal{X} has the following splitting

$$N_{\mathcal{Z}} = N_1 \oplus N_2 \oplus N_3 \oplus N',$$

where $N_i = N_{\mathcal{Z}|\mathcal{X}_{(g_i)}}$ are the normal bundles of \mathcal{Z} in $\mathcal{X}_{(g_i)}$ and N' is defined by the equation. $N_i, i = 1, 2, 3$ and N' are further decomposed into line eigenbundles

$$N_i = \oplus_{j=1}^{k_i} L_{ij}; N' = \oplus_{j=1}^k L'_j.$$

The splitting of normal bundle $N_{\mathcal{X}_{(g_i)}|\mathcal{X}}$ restricting to \mathcal{Z} is compatible with this splitting. For instance,

$$N_{\mathcal{X}_{(g_1)}|\mathcal{X}}|_{\mathcal{Z}} = N_2 \oplus N_3 \oplus N' = \left(\oplus_{j=1}^{k_2} L_{2j} \right) \oplus \left(\oplus_{j=1}^{k_3} L_{3j} \right) \oplus \left(\oplus_{j=1}^k L'_j \right)$$

and correspondingly, near \mathcal{Z}

$$t(g_1) = t_2(g_1)t_3(g_1)t'(g_1).$$

The terms in the right hand side above are defined in the obvious way. Similarly,

$$t(g_2) = t_1(g_2)t_3(g_2)t'(g_2) \text{ and } t(g_3) = t_1(g_3)t_2(g_3)t'(g_3).$$

We look at the *formal* expression

$$(3.6) \quad \frac{t(g_1) \wedge t(g_2)}{t(g_3)} = \frac{t_2(g_1)t_1(g_2)}{t_1(g_3)t_2(g_3)} \{t_3(g_1)t_3(g_2)\} \frac{t'(g_1)t'(g_2)}{t'(g_3)}.$$

- (1) Note that $(g_1, g_2, g_3^{-1}) \in T_o^3$, then it's easy to check that the first fraction simplifies to 1 when restricted to \mathcal{Z} .
- (2) In light of the equations (3.1) and (3.2), the term $\{t_3(g_1)t_3(g_2)\}$ is a Thom form τ_3 of N_3 , representing the Poincaré dual of \mathcal{Z} in $\mathcal{X}_{(g_3)}$.
- (3) To see what happens to $\frac{t'(g_1)t'(g_2)}{t'(g_3)}$, let us look at each L'_j . Suppose g_i acts on L'_j as multiplication of $e^{2\pi i \theta_{ij}}$ and the Thom form of L'_j is $[l'_j]$, then the exponent of $[l'_j]$ in $t(g_i)$ is θ_{ij} . Now $g_3 = g_1g_2$ implies that $\theta_{1j} + \theta_{2j}$ is either θ_{3j} or $\theta_{3j} + 1$. Hence,

$$\frac{[l'_j]^{\theta_{1j}} [l'_j]^{\theta_{2j}}}{[l'_j]^{\theta_{3j}}} = \begin{cases} 1, & \text{if } \theta_{1j} + \theta_{2j} = \theta_{3j}, \\ [l'_j], & \text{if } \theta_{1j} + \theta_{2j} = \theta_{3j} + 1. \end{cases}$$

The right hand side of the above becomes ordinary forms when restricted to \mathcal{Z} . Set

$$\Theta_{(g_1, g_2)} = \frac{t'(g_1)t'(g_2)}{t'(g_3)} \Big|_{\mathcal{Z}},$$

then $\Theta_{(g_1, g_2)}$ is an ordinary form. In fact, it's quite clear that $[\Theta_{(g_1, g_2)}] = e(E'_{(g_1, g_2)}) \in H^*(\mathcal{Z})$ where

$$(3.7) \quad E'_{(g_1, g_2)} = \bigoplus_{\theta_{1j} + \theta_{2j} = \theta_{3j} + 1} L'_j.$$

It follows that (3.6) gives an honest form on $\mathcal{X}_{(g_3)}$ and $a_3 = i_{(g_3)}(\alpha_3)$ where

$$[\alpha_3] = [i_1^*(\alpha_1) \wedge i_2^*(\alpha_2) \wedge \Theta_{(g_1, g_2)}] \wedge \tau_3 \in H^*(\mathcal{X}_{(g_3)}),$$

is given by $[i_1^*(\alpha_1) \wedge i_2^*(\alpha_2) \wedge \Theta_{(g_1, g_2)}] \in H^*(\mathcal{Z})$ via Thom homomorphism. \square

Corollary 3.3. *The product \wedge defines an associative ring structure on $H_{CR}^*(\mathcal{X})$.*

Proof. Assuming all intersections in the following has only one component. The general case is dealt with similarly as in the proposition. The equation we need to establish is:

$$\begin{aligned} (a_1 \wedge a_2) \wedge a_3 &= i_{(g_5)} \left\{ (i_4^* \{ (i_1^*(\alpha_1) \wedge i_2^*(\alpha_2) \wedge \Theta_{(g_1, g_2)}) \wedge \tau_4 \} \wedge i_3^*(\alpha_3) \wedge \Theta_{(g_4, g_3)}) \wedge \tau_5 \right\} \\ &= a_1 \wedge (a_2 \wedge a_3) = i_{(g_5)} \left\{ (i_1^*(\alpha_1) \wedge i_{4'}^* \{ (i_2^*(\alpha_2) \wedge i_3^*(\alpha_3) \wedge \Theta_{(g_2, g_3)}) \wedge \tau_{4'} \} \wedge \Theta_{(g_1, g_{4'})}) \wedge \tau_5 \right\}, \end{aligned}$$

where on the left hand side $\mathcal{X}_{(g_1)} \cap \mathcal{X}_{(g_2)} = \mathcal{Z}_4 \subset \mathcal{X}_{(g_4)}$ with $(g_1, g_2, g_4^{-1}) \in T_o^3$ and $\mathcal{X}_{(g_4)} \cap \mathcal{X}_{(g_3)} = \mathcal{Z}_5 \subset \mathcal{X}_{(g_5)}$ with $(g_4, g_3, g_5^{-1}) \in T_o^3$ and on the right hand side $\mathcal{X}_{(g_2)} \cap \mathcal{X}_{(g_3)} = \mathcal{Z}_{4'} \subset \mathcal{X}_{(g_{4'})}$ with $(g_1, g_2, g_{4'}^{-1}) \in T_o^3$ and $\mathcal{X}_{(g_{4'})} \cap \mathcal{X}_{(g_3)} = \mathcal{Z}_5 \subset \mathcal{X}_{(g_5)}$ with $(g_{4'}, g_3, g_5^{-1}) \in T_o^3$. The notation " \subset " denotes embedding given by the composition of arrows. The rest of the notations are as in the proposition. Let $\mathcal{Z} = \mathcal{X}_{(g_1)} \cap \mathcal{X}_{(g_2)} \cap \mathcal{X}_{(g_3)}$ then both sides of the equation is supported in a neighbourhood of \mathcal{Z} . We rewrite the left hand side:

$$LHS = i_{(g_5)} \left\{ ((i_1^*(\alpha_1) \wedge i_2^*(\alpha_2) \wedge i_3^*(\alpha_3)) \wedge (i_4^*(\Theta_{(g_1, g_2)}) \wedge \Theta_{(g_4, g_3)})) \wedge (i_4^*(\tau_4) \wedge \tau_5) \right\}$$

where i_\bullet is the inclusion of \mathcal{Z} in $\mathcal{X}_{(g_\bullet)}$ for $\bullet = 1, 2, 3$ and i_4 is the inclusion of \mathcal{Z}_5 in $\mathcal{X}_{(g_4)}$. Then $(i_4^*(\tau_4) \wedge \tau_5) = \tau_{\mathcal{Z}}(\mathcal{X}_{(g_5)})$. It follows from (3.7) that $(i_4^*(\Theta_{(g_1, g_2)}) \wedge \Theta_{(g_4, g_3)})$ represents the Euler class of the following bundle over \mathcal{Z} :

$$E'_{(g_1, g_2, g_3)} = \bigoplus_{\theta_{1j} + \theta_{2j} + \theta_{3j} = \theta_{5j} + 2} E_j,$$

where E_j 's are the complex line bundles appearing in the decomposition (3.4) for $\mathcal{X}_{(g_1, g_2, g_3)}$. Thus we have:

$$LHS = i_{(g_5)} \left\{ ((i_1^*(\alpha_1) \wedge i_2^*(\alpha_2) \wedge i_3^*(\alpha_3)) \wedge e(E'_{(g_1, g_2, g_3)})) \wedge \tau_{\mathcal{Z}}(\mathcal{X}_{(g_5)}) \right\}$$

The equation can then be shown by similar rewriting of the right hand side. \square

3.4. Obstruction bundle and obstruction form. The natural map $T^2 \rightarrow T_o^3 : ((g, h)) \mapsto ((g, h, (gh)^{-1}))$ is an isomorphism and we have isomorphism $\tilde{\mathcal{X}}^2 \rightarrow \tilde{\mathcal{X}}_o^3$ correspondingly. We consider a component $\mathcal{Z}_j = \mathcal{X}_{((h_{1,j}, h_{2,j}))}$ as in the proof of proposition 3.2 and use \mathcal{Z}_j to denote also the 3-multisector $\mathcal{X}_{((h_{1,j}, h_{2,j}, g_{3,j}^{-1}))}$, in the notation of the previous section. We show that

Proposition 3.4. $\Theta_{(h_{1,j}, h_{2,j})} = e(E_{((h_{1,j}, h_{2,j}, g_{3,j}^{-1}))})$ on \mathcal{Z}_j .

Proof. As we are only considering one component, we let $g_1 = h_{1,j}$, $g_2 = h_{2,j}$, $g_3 = g_{3,j}$ and $\mathcal{Z} = \mathcal{Z}_j$. It then suffices to show that $E'_{(g_1, g_2)} \cong E_{(\mathbf{g})}$ on \mathcal{Z} . We'll use the notations in §2.5.

Let $e : \mathcal{Z} \rightarrow \mathcal{X}$ be the evaluation map. With decomposition (3.4) for \mathcal{Z} and the almost complex structure on \mathcal{X} , the matrices representing the action of elements in $\langle \mathbf{g} \rangle$ can all be diagonalized. In particular we have

$$g_i = \text{diag}(1, \dots, 1, e^{2\pi i \theta_{i1}}, \dots, e^{2\pi i \theta_{im}}), \text{ where } \theta_{ij} \in \mathbb{Q} \cap [0, 1), \text{ for } i = 1, 2, 3.$$

The fiber of $E_{(\mathbf{g})}$ at y is then

$$\begin{aligned} (3.8) \quad E_{(\mathbf{g}), y} &= (H^{0,1}(\Sigma) \otimes T_{e(y)} G_0)^{\langle \mathbf{g} \rangle} \\ &= (H^{0,1}(\Sigma) \otimes T_{\tilde{y}} \mathcal{S}_{(\mathbf{g})})^{\langle \mathbf{g} \rangle} \oplus \bigoplus_{j=1}^m (H^{0,1}(\Sigma) \otimes E_j|_y)^{\langle \mathbf{g} \rangle} \\ &= H^1(S^2, (\phi_*(T_{\tilde{y}} \mathcal{S}_{(\mathbf{g})}))^{\langle \mathbf{g} \rangle}) \oplus \bigoplus_{j=1}^m H^1(S^2, (\phi_*(E_j|_y))^{\langle \mathbf{g} \rangle}) \end{aligned}$$

where $\phi : \Sigma \rightarrow S^2$ is the branched covering, ϕ_* is the push-forward of the constant sheaves. Let V be $\langle \mathbf{g} \rangle$ -vector space of (complex) rank v and $m_{i,j} \in \mathbb{Z} \cap [0, r_i)$ be the weights of action of g_i on V . Applying the index formula (proposition 4.2.2 in [3]) to $(\phi_*(V))^{\langle \mathbf{g} \rangle}$ we have

$$\chi = v - \sum_{i=1}^3 \sum_{j=1}^v \frac{m_{i,j}}{r_i},$$

Here we use the fact that $c_1(\phi_*(V)) = 0$ for constant sheaf V . If $\langle \mathbf{g} \rangle$ action is trivial on V then $\chi = v$. For $V = E_j|_y$, we see that $v = 1$ and $\frac{m_{i+1}}{r_i}$ is just θ_{ij} .

With the above preparations, we have the following

- (1) $(H^{0,1}(\Sigma) \otimes T_{\bar{y}}\mathcal{S}_{(\mathbf{g})})^{\langle \mathbf{g} \rangle} = \{0\}$ and
- (2) $(H^{0,1}(\Sigma) \otimes E_j|_y)^{\langle \mathbf{g} \rangle}$ is nontrivial (\Rightarrow rank 1) $\iff \sum_{i=1}^3 \theta_{ij} = 2$. (Note that this sum is either 1 or 2.) Moreover, it is clear that

$$(H^{0,1}(\Sigma) \otimes E_j|_y)^{\langle \mathbf{g} \rangle} \cong E_j|_y.$$

It follows that

$$E_{(\mathbf{g})} = \bigoplus_{\sum_{i=1}^3 \theta_{ij}=2} E_j,$$

which obviously matches with (3.7) (since the θ_{3j} here is $1 - \theta_{3j}$ in (3.7)). \square

3.5. Ring isomorphism. So far, on $H_{CR}^*(\mathcal{X})$ we have two different product structures: Chen-Ruan product “ \cup ” and wedge product “ \wedge ”. We have

Theorem 3.5. $(H_{CR}^*(\mathcal{X}), \cup) \cong (H_{CR}^*(\mathcal{X}), \wedge)$ as rings.

Proof. Let α, β and γ be as in §2.6. We show that

$$\langle \alpha \cup \beta, \gamma \rangle = \int_{\mathcal{X}}^{orb} i_{(g_1)}(\alpha) \wedge i_{(g_2)}(\beta) \wedge i_{(g_3)}(\gamma).$$

The right hand side is

$$\begin{aligned} \int_{\mathcal{X}}^{orb} i_{(g_1)}(\alpha) i_{(g_2)}(\beta) i_{(g_3)}(\gamma) &= \int_{\mathcal{X}}^{orb} \pi_1^*(\alpha) \pi_2^*(\beta) \pi_3^*(\gamma) \prod_{i=1}^3 t(g_i) \\ &= \int_{\mathcal{X}}^{orb} \pi_1^*(\alpha) \pi_2^*(\beta) \pi_3^*(\gamma) \prod_{j=1}^m [l_j]^{\sum_{i=1}^3 \theta_{ij}} \\ &= \int_{\mathcal{X}}^{orb} \pi_1^*(\alpha) \pi_2^*(\beta) \pi_3^*(\gamma) \Omega(X_{(\mathbf{g})}) \prod_{j=1}^m [l_j]^{\sum_{i=1}^3 \theta_{ij} - 1} \\ &= \int_{\mathcal{X}_{(\mathbf{g})}}^{orb} e_1^*(\alpha) e_2^*(\beta) e_3^*(\gamma) \Theta_{(g_1, g_2)} \\ &= \int_{\mathcal{X}_{(\mathbf{g})}}^{orb} e_1^*(\alpha) e_2^*(\beta) e_3^*(\gamma) e(E_{(\mathbf{g})}). \end{aligned}$$

Here $\Omega(X_{(\mathbf{g})})$ is the Thom form of $X_{(\mathbf{g})}$ in X , which represents the Poincaré dual of $X_{(\mathbf{g})}$. The theorem then follows from definition of Chen-Ruan orbifold cup product (§2.6). \square

We therefore constructed a deRham type model of $H_{CR}^*(\mathcal{X})$. The advantages with this formulation is two-fold. Firstly, the product on Chen-Ruan orbifold cohomology can now be given directly. Secondly, as shown in the proof of the theorem, when computing the three point functions, the domain of integration are unified to be \mathcal{X} . This will make it easier for application.

4. SYMPLECTIC REDUCTION FOR TORUS ACTION AND WALL CROSSING

As an application of our deRham model we consider symplectic reduction for torus action. Let $G = T^l$, (M, ω) be a $2N$ -dimensional symplectic manifold with a Hamiltonian G action. Let the moment map be $\mu : M \rightarrow \mathfrak{g}^*$, $p \in \mathfrak{g}^*$ lying in the image of μ be a regular value and $M(p) = \mu^{-1}(p)$. Then it's well known that $X_p = M//_p G = M(p)/G$ is a symplectic orbifold of dimension $2n = 2(N - l)$. It is known that there is a chamber structure on \mathfrak{g}^* such that X_p and X_q are diffeomorphic when p and q are in a same chamber C . It would be interesting

to investigate how the orbifold cohomology differs when p and q are in different chambers. In this section, we will give a wall crossing formula for the 3-point function. As one expects, the difference of 3-point functions on X_p and X_q is contributed by fixed loci of the G action on M . With the original formulation given in §2.1, it is not easy to write a clean wall crossing formula due to the appearance of twisted sectors and obstruction forms. The new formulation then has an advantage in dealing with these issues, at least at the level of presentation.

4.1. Orbi-structure of X_p . Let $\pi : M(p) \rightarrow X_p$ be the quotient map. Let $x \in X_p$ and $\tilde{x} \in \pi^{-1}(x)$, then a local orbifold chart U near x is given by a normal slice at \tilde{x} of the orbit $G \circ \{\tilde{x}\}$ in $M(p)$, where G_x is the isotropy group at \tilde{x} . Since G is abelian, we are in the situation discussed in §2.1. The local group G_x is a finite subgroup of G and we make the following non-essential assumption to simplify notations.²

Assumption 4.1. *For any finite subgroup H of G , the points of M which are fixed by H have at most one component over any $p \in \mathfrak{g}^*$.*

Under this assumption, the labeling set T^k for k -multisectors of X_p is subset of G^k . Let $\mathbf{g} = (g_1, \dots, g_k) \in G_x^k$, $\langle \mathbf{g} \rangle$ be the subgroup of G_x generated by \mathbf{g} , $M^{(\mathbf{g})}$ be the fixed point set of $\langle \mathbf{g} \rangle$ in M and $M(p)^{(\mathbf{g})} = M^{(\mathbf{g})} \cap M(p)$. Then we have

$$(X_p)_{(\mathbf{g})} = M(p)^{(\mathbf{g})} / G.$$

4.2. Equivariant set-up on M . For simplicity, we will assume $G = S^1$. Let \mathcal{F} be the set of fixed points of S^1 . For $g \in G$, define M^g to be the submanifold in M fixed by g . The interesting case is that $M^g - \mathcal{F} \neq \emptyset$. From now on, we always assume that this is the case.

The G action gives a G -equivariant decomposition

$$TM|_{M^g} = \bigoplus_{j=1}^m \tilde{E}_j \oplus TM^g.$$

This decomposition descends to the one in (2.2) with TM^g further splits into $\mathbb{R} \oplus TM(p)^g$ on $M(p)$. Let $[\tilde{l}_j]$ be an equivariant Thom class for \tilde{E}_j supported in an equivariant neighbourhood of 0-section of \tilde{E}_j . Let θ_j be the weights of g action on fiber of \tilde{E}_j then

Definition 4.1. Equivariant twist factor for M^g is the formal equivariant form:

$$[\tilde{t}(g)] = \prod_{j=1}^m [\tilde{l}_j]^{\theta_j}.$$

As before, formally we have $\tilde{t}(g) \in H_G^{2\iota(g)}(M)$. We then make the following definitions parallel to those in §3.1

$$\tilde{i}_{(g)} : H_G^*(M^g) \rightarrow H_G^*(M) : \tilde{\alpha} \mapsto \tilde{i}_{(g)}(\tilde{\alpha}) = \tilde{\pi}^*(\tilde{\alpha})\tilde{t}(g), \text{ and}$$

$$H_{G,CR}^*(M) = \bigoplus_{(g) \in G} H_G^*(M^g)$$

with the degree shifting given by $2\iota_{(g)}$.

The Kirwan map for the usual (equivariant) cohomology is defined for regular value p of the moment map μ as following:

$$\kappa_p : H_G^*(M) \xrightarrow{i_p^*} H_G^*(M(p)) \xrightarrow{\cong} H^*(X_p).$$

Kirwan surjectivity ([7]) states that κ_p is surjective when M is compact. For some cases of non-compact M , e.g. \mathbb{C}^n with linear actions, the Kirwan map is also surjective. Suppose κ_p is surjective for M as well as M^g for all $g \in G$ and define

$$\kappa_p : H_{G,CR}^*(M) \rightarrow H_{CR}^*(X_p)$$

by the direct sum on the factors, then the following is obvious:

²Otherwise the labeling set T_k below would have to take into account different components of the points fixed by subgroup H since the same elements in local groups for different component are *not* equivalent, which only leads to messier notations.

Proposition 4.2. *The Kirwan map κ_p is surjective.* \square

4.3. Wall crossing of Chen-Ruan orbifold cup product. The set of regular values of μ consists of points outside a collection of hyperplanes in \mathfrak{g}^* . The codimension 1 hyperplanes are called *walls* of the moment map. Let W be such a wall and let $\xi_1 \in \mathfrak{g}$ be a primary vector such that

$$W \subset \{v \in \mathfrak{g}^* | \langle \xi_1, v \rangle = 0\}.$$

Extend ξ_1 to a basis $\{\xi_1, \dots, \xi_l\}$ of \mathbb{Z}^l -lattice of \mathfrak{g} and fix the basis in the following. Let H be the subgroup generated by ξ_1 and H' be generated by $\{\xi_2, \dots, \xi_l\}$ be its complement. Let $\{u_i\}$ be the dual basis of $\{\xi_i\}$. Suppose $p \in \text{Image}(\mu)$ be a regular value and $a \in \mathbb{R}^+$ small such that $q = p + au_1$ are in different chambers separated by W . Let $I = [p, q]$ denote the line segment between p and q .

Let $X_q = M//_q G = M(q)/G$ and $\tilde{M} = \mu^{-1}(I)$. Let $\mathbf{g} = (g_1, g_2, g_3)$ such that $g_1 g_2 g_3 = 1$ and $\alpha_p \in H^*(X_{(g_1)})$, $\beta_p \in H^*(X_{(g_2)})$ and $\gamma_p \in H^*(X_{(g_3)})$ with $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma} \in H_{G, CR}^*(M)$ be their equivariant lifting respectively. Let $\alpha_q = \kappa_q(\tilde{\alpha})$ and so on. We may arrange them into the following diagram:

$$\begin{array}{ccc} & H_{G, CR}^*(M) & \\ \swarrow \kappa_p & & \searrow \kappa_q \\ H_{CR}^*(X_p) & & H_{CR}^*(X_q) \end{array}$$

Let $F = \cup_j F_j \subset \tilde{M}$ be the fixed point set of H action. Then our main theorem is that the contribution to the difference between $\langle \alpha_p \cup \beta_p, \gamma_p \rangle$ and $\langle \alpha_q \cup \beta_q, \gamma_q \rangle$ is localized at F . To simplify notations, we state theorem for the case $G = H = S^1$.

Theorem 4.3. *Suppose $G = S^1$, $\alpha_p, \beta_p, \gamma_p, \alpha_q, \beta_q, \gamma_q$ and F_j are given as above. Then*

$$(4.1) \quad \langle \alpha_q \cup \beta_q, \gamma_q \rangle - \langle \alpha_p \cup \beta_p, \gamma_p \rangle = \sum_j \int_{F_j} \frac{\tilde{i}_{(g_1)}(\tilde{\alpha}) \tilde{i}_{(g_2)}(\tilde{\beta}) \tilde{i}_{(g_3)}(\tilde{\gamma})}{e_G(N_{F_j})},$$

where $e_G(N_{F_j})$ is the equivariant euler class of normal bundle N_{F_j} of F_j in M .

Proof. Note that, by theorem 3.5, we have

$$\langle \alpha_r \cup \beta_r, \gamma_r \rangle = \int_{X_r}^{orb} i_{(g_1)}(\alpha_r) i_{(g_2)}(\beta_r) i_{(g_3)}(\gamma_r)$$

for $r = p$ or q . Then (4.1) follows from the standard localization formula for the integration

$$\int_{\tilde{M}} \tilde{i}_{(g_1)}(\tilde{\alpha}) \tilde{i}_{(g_2)}(\tilde{\beta}) \tilde{i}_{(g_3)}(\tilde{\gamma}).$$

\square

5. EXAMPLES

5.1. Weighted projective spaces. Weighted projective space $\mathbb{P}(w_1, \dots, w_n)$ of (complex) dimension $n-1$ can be described as the symplectic quotient of a linear S^1 action ρ on $M = \mathbb{C}^n$, with weights $w_1, \dots, w_n \in \mathbb{Z}^+$ on the eigenspaces. Let $W = (w_1, \dots, w_n) \in \mathbb{Z}^n$ be the weight vector of the S^1 action. For simplicity, we assume that the greatest common divisor of w_i 's is 1. Let the basis $\{v_i\}$ of \mathbb{C}^n be given by the eigenvectors, then the moment map is given by

$$\mu : \mathbb{C}^n \rightarrow \mathbb{R} : \mu(z_1, \dots, z_n) = \frac{1}{2} \sum_i w_i |z_i|^2.$$

0 is the only singular value which is the wall and $\mu^{-1}(0) = \{0\} \in \mathbb{C}^n$ is the unique fixed point. Let $p < 0$ and $q > 0$ then we have $X_p = \emptyset$ and $X_q = \mathbb{P}(w_1, \dots, w_n)$ with scaled symplectic (or

Kähler) form. It follows that in (4.1) there is only one term on either side and $e_G(N_F)$ here is simply $u^n \prod_i w_i$. Furthermore, the twisted sectors are copies of lower dimensional weighted projective subspaces with weights $(w_{i \in I})$ for some $I \subset \{1, \dots, n\}$ and we denote them $\mathbb{P}_I(W)$. Thus we have

$$(5.1) \quad \langle \alpha_1 \cup \alpha_2, \alpha_3 \rangle = \left(\frac{\tilde{i}_{(g_1)}(\tilde{\alpha}_1) \tilde{i}_{(g_2)}(\tilde{\alpha}_2) \tilde{i}_{(g_3)}(\tilde{\alpha}_3)}{u^n \prod_i w_i} \right) \Big|_{z=0},$$

for $\alpha_i \in X_{(g_i)} \cong \mathbb{P}_{I_i}(W)$ and $g_1 g_2 g_3 = 1$. The evaluation at $z = 0$ implies that only the terms with no form part contribute in the various equivariant twisted forms.

Let's apply the formula (5.1) to $X = \mathbb{P}(W)$ where $W = (1, 2, 2, 3, 3, 3)$, which is studied in [5]. Let $g = \omega$ the 3-rd root of 1, then the twisted sector $X_{(g)}$ of X defined by g is isomorphic to $\mathbb{P}(3, 3, 3)$, or equivalently, \mathbb{P}^2 with trivial \mathbb{Z}_3 action. It's straight forward to see that $\iota_{(g)} = \frac{1}{3} + \frac{2}{3} + \frac{2}{3} = \frac{5}{3}$. Let $\alpha_i \in H^*(X_{(g)})$ for $i = 1, 2, 3$, then in order for $\langle \alpha_1 \cup \alpha_2, \alpha_3 \rangle \neq 0$, we must have $\alpha_i \in H^0(X_{(g)})$. Without loss of generality, let $\alpha_i = 1_{(g)}$. Then applying (5.1) we have

$$\begin{aligned} \langle 1_{(g)} \cup 1_{(g)}, 1_{(g)} \rangle &= \left(\frac{(\tilde{t}_{(g)})^3}{u^5 \prod_i w_i} \right) \Big|_{z=0}, \\ &= \left(\frac{\left((u + \dots)^{\frac{1}{3}} (2u + \dots)^{\frac{2}{3}} (2u + \dots)^{\frac{2}{3}} \right)^3}{2^2 3^3 u^5} \right) \Big|_{z=0} \\ &= \frac{4}{27} \end{aligned}$$

where \dots stands for terms evaluating to 0 when $z = 0$. This verifies the computation in [5].

5.2. Mirror quintic orbifolds. We consider the mirror quintic orbifold Y , which is defined as a generic member of the anti-canonical linear system in the following quotient of \mathbb{P}^4 by $(\mathbb{Z}_5)^3$:

$$[z_1 : z_2 : z_3 : z_4 : z_5] \sim [\xi^{a_1} z_1 : \xi^{a_2} z_2 : \xi^{a_3} z_3 : \xi^{a_4} z_4 : \xi^{a_5} z_5],$$

where $\sum a_i \equiv 0 \pmod{5}$ and $\xi = e^{\frac{2\pi i}{5}}$. Let \triangle° be the polytope with vertices $v_0 = e_0 = (-1, -1, -1, -1)$ and $v_i = e_0 + 5e_i$ for $i = 1, 2, 3, 4$ where $\{e_i\}_{i=1}^4$ is the standard basis of \mathbb{R}^4 . Then coning the faces of \triangle° gives the fan Σ which defines $X = \mathbb{P}^4/(\mathbb{Z}_5)^3$. We can also obtain Y as the quotient by the above $(\mathbb{Z})^3$ of a quintic of the following form:

$$\tilde{Y} = \{z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + \psi z_1 z_2 z_3 z_4 z_5 = 0\}, \text{ where } \psi^5 \neq -5^5.$$

The computation for mirror quintic was first done in [8].

The ordinary cup product on Y is computed in [8] §6 and we refer to there for details. We also follow [8] §5 for the description of twisted sectors of Y . The twisted sectors of Y are either points or curves. The main simplification in applying our method is to compute the contribution from twisted sectors which are curves. Let $Y_{(\mathbf{g})}$ be a triple twisted sector which is an orbifold curve, where $(\mathbf{g}) = (g_1, g_2, g_3)$. Such curve only occurs as intersection of Y with some 2-dimensional invariant variety of X . It follows then the isotropy group for generic point in $Y_{(\mathbf{g})}$ can only be $G \cong \mathbb{Z}_5$ and we have $g_i \in G$. Furthermore, under the evaluation maps to Y , $Y_{(g_i)}$ and $Y_{(\mathbf{g})}$ have the same images, which we'll denote as $Y_{(G)}$.

Using the deRham model, we note that the formal maps

$$i_{(\cdot)} : H^*(Y_{(\cdot)}) \rightarrow H_{CR}^{*+\iota_{(\cdot)}}(Y)$$

where \cdot is one of g_i or \mathbf{g} , all factor through a tubular neighbourhood of $Y_{(G)}$ in Y . Since Y is orbifold Calabi-Yau, the degree shifting $\iota_{(\cdot)}$ is always non-negative integer. In particular, if

$g_i \neq id \in G$, we have to have $\iota_{(g_i)} = 1$. Let $\alpha_i \in H^*(Y_{(g_i)})$ and we consider the Chen-Ruan cup product $\alpha_1 \cup \alpha_2$. It suffices to evaluate the non-zero pairing of the following form

$$(5.2) \quad \langle \alpha_1 \cup \alpha_2, \alpha_3 \rangle = \int_Y^{ord} \wedge_{i=1}^3 i_{(g_i)}(\alpha_i) \neq 0.$$

When $g_3 = id$, we see that the Chen-Ruan cup product reduces to (ordinary) Poincaré duality. When $g_i \neq id$ for $i = 1, 2, 3$, by direct degree checking we find that $\alpha_i \in H^0(Y_{(g_i)})$ for all i and the wedge product in (5.2) is a multiple of product of twist factors $t(g_i) = [l_1]^{\theta_{i1}} [l_2]^{\theta_{i2}}$, where $[l_j]$'s are the Thom classes of the line bundle factors of the normal bundle. Without loss of generality, let $\alpha_i = 1_{(g_i)}$. Since $g_i \neq id$ by assumption, we have $\theta_{ij} > 0$ for all i, j . Thus

$$(5.3) \quad \int_Y^{ord} \wedge_{i=1}^3 i_{(g_i)}(1_{(g_i)}) = \int_{Y_{(G)}}^{orb} c.,$$

where $c.$ stands for the Chern class corresponding to either $[l_1]$ or $[l_2]$. Let X_2 be the 2 dimensional invariant subvariety of $X = \mathbb{P}^4/(\mathbb{Z}_5)^3$ such that $Y \cap X_2 = Y_{(G)}$. Then there are 2 invariant subvarieties $X_{3,1}$ and $X_{3,2}$ of dimension 3 which contains X_2 . Let $Y_j = Y \cap X_{3,j}$ for $j = 1, 2$. Then c_j above is simply the Chern class of the normal bundle of $Y_{(G)}$ in Y_j . To finish the computation, we note that the whole local picture can be lifted to $\tilde{Y} \subset \mathbb{P}^4$ where the Chern classes corresponding to c_j obviously integrate to 5. Then the quotient by $(\mathbb{Z}_5)^3$ gives the answer to the integration (5.3) as

$$\int_{Y_{(G)}}^{orb} c. = \frac{5}{125} = \frac{1}{25},$$

which verifies the computation in [8].

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